PCT-Invariance of Generalized de Broglie-Bargmann-Wigner Equations

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Generalized de Broglie-Bargmann-Wigner (GBBW)-equations are part of the derivation of effective theories for composite particles based on a nonlinear spinor field model with canonical quantization, relativistic invariant selfregularization and probability interpretation. Owing to the close connection between the model itself and its GBBW-equations the behavior of both is studied under *PCT*-transformations leading to the proof of invariance against such tranformation of the field Hamiltonian as well as of the GBBW-equations. Finally the algebraic formulation of the operators of these transformations in general state spaces is derived as the nonperturbative algebraic treatment of the model is referred to such spaces which are inequivalent to the Fock space representation of perturbation theory.

Key words: Discrete Transformations; Relativistic Many-body Equations; Composite Elementary Particles.

1. Introduction

Generalized de Broglie-Bargmann-Wigner (GBBW)-equations are relativistically invariant quantum mechanical many body equations with nontrivial interaction, selfregularization and probability interpretation. In previous papers the effect of continuous transformations and of permutation operations upon these equations and their solutions were studied in detail, [1 – 4]. In this paper we analyze the effect of discrete transformations, namely *PCT*-transformations upon these equations in order to complete the information about the physical meaning of these equations themselves as well as of their solutions.

As the GBBW-equations are part of a quantum field theoretic treatment of relativistic composite particle dynamics, it is advisable to study the transformation properties of this field theoretic model with respect to discrete transformations first. Because, if this field theoretic model was not *PCT*-invariant, it is hardly to imagine that the GBBW-equations could possess this invariance property.

Thus our investigation will be started by analyzing *PCT*-invariance of the abstract algebraic formulation of the theory and continued by considering the *PCT*-invariance in physically appropriate representation spaces. The latter is equivalent to the study of the effect of *PCT*-transformations on GBBW-equations and their solutions.

In pursuing this program discrete transformations of operators and of GBBW-equations have to be clearly distinguished. Field operators and corresponding quantitites such as field equations, Lagrangian densities, etc., are subjected to unitary or antiunitary transformations which in consequence produce discrete coordinate transformations, while GBBW-equations are directly transformed in configuration space in analogy to the treatment of the quantum mechanically Dirac equation and the treatment of matrix elements.

In literature discrete transformations are treated in nearly any textbook on quantum field theory. Unfortunately these treatments differ considerably in notations, formulation of the transformations and the corresponding theoretical arguments. So no unique way of representing this topic is established. In the following we refer to [5 – 7], and [8] where details about discrete transformations are given. As far as results are borrowed from literature we refer to the corresponding sources by giving pages and numbers of equations to allow an easy verification. The term *PCT* we adopt from [6].

In view of the various different notations we define the meaning of the symbols which are used in this paper in order to prevent misunderstandings of our formulations. Let ψ be a quantum mechanical wave function (classical spinor field) or a spinor field operator, respectively, then we define

ψ^* := Complex conjugation of classical fields and operator fields,
ψ^T := Transposition of classical spinor fields from column vector into row vector and vice versa,
$\psi^\dagger$:= Complex conjugation and transposition of classical spinor fields,
ψ^+: =Complex conjugation and transposition in spinspace as well as in state space of spinor field operators := Hermitian conjugation,
$\bar{\psi}$:= $\psi^\dagger\gamma^0$:= classical adjoint spinor field,
$\bar{\psi}$:= $\psi^+\gamma^0$:= adjoint spinor field operator.

In former papers the *n*-fold transposition and complex conjugation of classical *n*-th order spin tensors φ which correspond to matrix elements of monomials of spinor field operators was denoted by φ^+ in accordance with [5] for spin tensors of degree *n* = 1, i. e., Dirac spinors. But in view of the above definitions this operation should be denoted by $\varphi^\dagger$.

2. *PCT*-invariance of the Spinorfield Model

From a physical point of view the corresponding field theoretic model can be considered as a quantum field theoretic generalization of de Broglie's fusion theory, [9], and as a mathematical realization and physical modification of Heisenberg's approach, [10]. In particular the model is based on a nonperturbatively regularized nonlinear spinor field Lagrangian with canonical quantization, relativistic invariance and probability interpretation. In addition, owing to the fusion idea, for its evaluation a new nonperturbative method for the treatment of composite particle dynamics was developed, see [11].

In the course of this development charge conjugated spinor fields were introduced instead of adjoint spinor fields. While, undoubtedly, the field Lagrangians are equivalent in both versions, the difference between these two formulations of the theory arises in the construction of the corresponding representation spaces, the elements of which are formed by the eigenstates of GBBW-equations.

In a first step we treat the transformation properties of spinor field operators in the conventional version, i. e., by using spinor fields and adjoint spinor fields. The model under consideration is defined by the Lagrangian density, see [11], Eq. (2.52):

$$\mathcal{L}(x) := \sum_{i=1}^{3} \lambda_i^{-1} \bar{\psi}_{A\alpha i}(x)(i\gamma^\mu\partial_\mu - m_i)_{\alpha\beta}\delta_{AB}\psi_{B\beta i}(x) - \frac{1}{2} g \sum_{h=1}^{2} \delta_{AB}\delta_{CD} v^h_{\alpha\beta} v^h_{\gamma\delta} \cdot \sum_{i,j,k=1}^{3} \bar{\psi}_{A\alpha i}(x)\psi_{B\beta j}(x)\bar{\psi}_{C\gamma j}(x)\psi_{D\delta l}(x). \tag{1}$$

The field operators are assumed to be Dirac spinors with index α = 1, 2, 3, 4 and additional isospin with index A = 1, 2 as well as auxiliary fields with index i = 1, 2, 3 for nonperturbative regularization.

The algebra of the field operators is defined by the anticommutators

$$[\psi^+_{A\alpha i}(\mathbf{r},t)\psi_{B\beta j}(\mathbf{r}',t)]_+ = \lambda_i\delta_{ij}\delta_{AB}\delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}') \tag{2}$$

resulting from canonical quantization. All other anticommutators vanish. The other quantities appearing in (1) are defined in detail in [11] and will be explicitly given in the course of the investigation. The transformation properties of the field operators for continuous groups were given in [1 – 4, 11]. Hence we can concentrate on the transformation properties for discrete groups. Owing to the conventional formulation of the spinor fields in (1), the transformation properties of Dirac fields for discrete transformations can be adopted from the literature.

For parity transformations *P* one obtains, see [8], Eq. (6.147) or [6], Eq. (8.41)

$$\begin{aligned} \mathcal{P}\psi_{A\alpha i}(\mathbf{r},t)\mathcal{P}^{-1} &= \eta^P\gamma^0_{\alpha\beta}\psi_{A\beta i}(-\mathbf{r},t), \\ \mathcal{P}\bar{\psi}_{A\alpha i}(\mathbf{r},t)\mathcal{P}^{-1} &= \eta^{P*}\bar{\psi}_{A\beta i}(-\mathbf{r},t)\gamma^0_{\beta\alpha}, \end{aligned} \tag{3}$$

where $\mathcal{P}$ is a unitary operator of space reflection in the algebraic state space. This operator and the other operators of discrete operations are explicitly constructed in Fock space in [5] and [12], while for more general state spaces their existence must be proved. As the algebraic treatment of our model refers to such more general state spaces it is necessary to secure the existence of these operators by the derivation of their explicit algebraic form. However, because in the course of our deductions the special algebraic form of these operators is not required, we postpone this proof to Section 6.

For charge conjugation one obtains, see [8], Eq. (6.163a)

$$\begin{aligned} \mathcal{C}\psi_{A\alpha i}(\mathbf{r},t)\mathcal{C}^{-1} &= \eta^C C_{\alpha\beta}\bar{\psi}_{A\beta i}(\mathbf{r},t)^T, \\ \mathcal{C}\bar{\psi}_{A\alpha i}(\mathbf{r},t)\mathcal{C}^{-1} &= \eta^C\psi_{A\beta i}(\mathbf{r},t)^T C_{\beta\alpha}, \end{aligned} \tag{4}$$

where $\mathcal{C}$ is the corresponding unitary operator in state space.

For time inversion the corresponding relations read, see [6], Eqs. (8.94), (8.100):

$$\begin{aligned}\mathcal{T}\psi_{A\alpha i}(\mathbf{r},t)\mathcal{T}^{-1} &= \eta^T(\gamma^5 C)_{\alpha\beta}\psi_{A\beta i}(\mathbf{r},-t),\\ \mathcal{T}\bar\psi_{A\alpha i}(\mathbf{r},t)\mathcal{T}^{-1} &= -\eta^{T*}\bar\psi_{A\beta i}(\mathbf{r},-t)(\gamma^5 C)_{\beta\alpha},\end{aligned} \tag{5}$$

where $\mathcal{T}$ is an antiunitary operator in state space.

Finally with an appropriate choice of the phase factors the combined action of *PCT*-transformations $\mathcal{A} := \mathcal{PCT}$ is given by, [6], Eq. (8.112)

$$\begin{aligned}\mathcal{A}\psi_{A\alpha i}(x)\mathcal{A}^{-1} &= (\gamma^5\gamma^0)_{\alpha\beta}\bar\psi_{A\beta i}(-x)^T,\\ \mathcal{A}\bar\psi_{A\alpha i}(x)\mathcal{A}^{-1} &= \psi_{A\beta i}(-x)^T(\gamma^5\gamma^0)_{\beta\alpha}.\end{aligned} \tag{6}$$

Equations (5) and (6) coincide with those of [8], Eqs. (6.151a) or (6.274), respectively, if the different definition of γ^5 in [6] and [8] is taken into account.

By means of these formulas the Lagrangian density (1) and the corresponding Hamiltonian density can be subjected to *PCT*-transformations.

We first treat the *PCT*-invariance of the action *W* resulting from (1) and afterwards show the invariance of *H* under *PCT*-transformations.

With specific values of the constants λ_i in (1), see [11], Eqs. (2.34), (2.51), the corresponding spinor theory based on (1) and (2) is automatically nonperturbatively self regularized. If one defines the averaged subfermion fields

$$\Psi_{A\alpha}(x) := \sum_{i=1}^{3}\psi_{A\alpha i}(x), \tag{7}$$

a consequence of selfregularization is that

$$[\bar\Psi_{A\alpha}(\mathbf{r},t)\Psi_{B\beta}(\mathbf{r}',t)]_+ = 0 \tag{8}$$

holds. And by means of (7) the Lagrangian density (1) can be rewritten in the form

$$\begin{aligned}\mathcal{L}(x) &= \sum_A\sum_{i=1}^{3}\lambda_i^{-1}\bar\psi_{A\alpha i}(x)(i\gamma^\mu\partial_\mu - m_i)_{\alpha\beta}\psi_{A\beta i}(x)\\ &-\frac{1}{2}g\sum_{A,C}\sum_{h=1}^{2}\bar\Psi_{A\alpha}(x)v^h_{\alpha\beta}\Psi_{A\beta}(x)\bar\Psi_{C\gamma}(x)v^h_{\gamma\delta}\Psi_{C\delta}(x)\\ &=: \mathcal{L}_0(x)+\mathcal{L}_I(x).\end{aligned} \tag{9}$$

From the special form of (9) it follows that this problem can be reduced to the *PCT*-transformations of bilinear expressions of field operators.

We first discuss the Dirac part $\mathcal{L}_0$ of (9). In order to adapt $\mathcal{L}_0$ to the form used in the literature we convert this term in the conventional fashion by writing

$$\begin{aligned}\mathcal{L}_0(x) \equiv \sum_A\sum_{i=1}^{3}\lambda_i^{-1}\bar\psi_{A\alpha i}(x)\\ \cdot\left(\frac{1}{2}i\gamma^\mu\overset{\leftrightarrow}{\partial}_\mu - m_i\right)_{\alpha\beta}\psi_{A\beta i}(x)\end{aligned} \tag{10}$$

with the definition

$$\bar\psi\overset{\leftrightarrow}{\partial}_\mu\psi := \bar\psi\overset{\rightarrow}{\partial}_\mu\psi - \bar\psi\overset{\leftarrow}{\partial}_\mu\psi. \tag{11}$$

In [6], Sects. 8.3 – 8.6, the discrete transformations of $\mathcal{L}_0(x)$ are carefully elaborated. Apart from a normal ordering constant which can be substracted and which is irrelevant for the dynamics, from these calculations it follows

$$\begin{aligned}\mathcal{A}\mathcal{L}_0(x)\mathcal{A}^{-1} = \sum_A\sum_{i=1}^{3}\lambda_i^{-1}\bar\psi_{A\alpha i}(-x)\\ \cdot\left(\frac{1}{2}i\gamma^\mu\overset{\leftrightarrow}{\partial}_\mu(-x) - m_i\right)_{\alpha\beta}\psi_{A\beta i}(-x) = \mathcal{L}_0(-x)\end{aligned} \tag{12}$$

with $\partial_\mu(-x) := -\partial/\partial x^\mu$.

The *PCT*-tranformation with respect to $\mathcal{L}_I$ is much easier to handle because of the vanishing anticommutator (8).

For $v^1 := \mathbf{1}$ and $v^2 := i\gamma^5$ one obtains

$$\mathcal{A}\bar\Psi(x)\Psi(x)\mathcal{A}^{-1} = -\bar\Psi(-x)\Psi(-x) \tag{13}$$

and

$$\mathcal{A}\bar\Psi(x)i\gamma^5\Psi(x)\mathcal{A}^{-1} = -\bar\Psi(-x)i\gamma^5\Psi(-x), \tag{14}$$

and therefore

$$\begin{aligned}\mathcal{A}\mathcal{L}_I(x)\mathcal{A}^{-1} = -\frac{1}{2}g\sum_{h=1}^{2}\mathcal{A}\bar\Psi_{A\alpha}(x)v^h_{\alpha\beta}\Psi_{A\beta}(x)\mathcal{A}^{-1}\\ \cdot\mathcal{A}\bar\Psi_{C\gamma}(x)v^h_{\gamma\delta}\Psi_{C\delta}\mathcal{A}^{-1} = \mathcal{L}_I(-x).\end{aligned} \tag{15}$$

Combining (9),(12) and (15) yields

$$\mathcal{A}\mathcal{L}(x)\mathcal{A}^{-1} = \mathcal{L}(-x) \tag{16}$$

for the spinor field Lagrangian density (1) or (9), respectively, in accordance with the *PCT*-theorem, see [6], Sect. 8.7, or [8], Sect. 6.10, respectively. Owing to

(12) the transformation (16) includes the corresponding transformation of the derivatives with respect to *x* too. This fact means: if one replaces $-x$ by $x' = -x$ one obtains the original Lagrangian referred to x'. This can be summarized into

Proposition 1: The spinor field Lagrangian density (1) is forminvariant under *PCT*-transformations.

Of course, as the Lagrangian density (9) generates a Poincare invariant local field theory, one could have immediately concluded that (16) must hold. But it was the purpose of the explicit deduction of (16) to show that for a spinor field Lagrangian with selfregularization no infinite counter terms for the interaction term are required in order to keep the transformation (16) finite. This can be expressed as

Proposition 2: For the application of *PCT*-transformations no normal ordering of the Lagrangian density $\mathcal{L}_I$ is required.

In conventional theories without selfregularization such normalordering is unavoidable.

Furthermore, from (16) it follows immediately that

$$W' = \int \mathcal{L}'(x)\mathrm{d}^4x = \int \mathcal{L}(-x)\mathrm{d}^4x = \int \mathcal{L}(x)\mathrm{d}^4x = W, \tag{17}$$

i. e., the action *W* of the spinor field theory is invariant under *PCT*-transformations.

Finally we treat the Hamiltonian of the spinor field. Its density is given by, see [11], eq. (2.57)

$$\mathcal{H}(x) := \sum_{i=1}^{3} \lambda_i^{-1}\bar{\psi}_{A\alpha i}(x)(-\frac{1}{2}i\gamma^k \overleftrightarrow{\partial}_k + m_i)_{\alpha\beta}\psi_{A\beta i}(x) - \mathcal{L}_I(x). \tag{18}$$

This density can be equivalently expressed by

$$\mathcal{H}(x) = \sum_{i=1}^{3} \lambda_i^{-1}\bar{\psi}_{A\alpha i}(x)\frac{1}{2}i\gamma^0 \overleftrightarrow{\partial}_0 \psi_{A\beta i}(x) - \mathcal{L}(x), \tag{19}$$

and the treatment of the Dirac Lagrangian $\mathcal{L}_0(x)$ under *PCT*-transformations given above, can be immediately applied to the first term on the right hand side of (19). Hence it follows

$$\mathcal{A}\mathcal{H}(x)\mathcal{A}^{-1} = \mathcal{H}(-x). \tag{20}$$

Then the Hamiltonian can be defined by

$$H := \int \mathcal{H}(x)\delta(n^\rho x_\rho)\mathrm{d}^4x, \tag{21}$$

where n^ρ is a timelike unit vector.

If one observes that

$$\delta(n^\rho x_\rho)\mathrm{d}^4x = \delta(n^\rho(-x_\rho))\mathrm{d}^4(-x) \tag{22}$$

holds, it follows with (20), (21) and $x' = -x$ that

$$\mathcal{A}H\mathcal{A}^{-1} = \int \mathcal{A}\mathcal{H}(x)\mathcal{A}^{-1}\delta(n^\rho x_\rho) = \int \mathcal{H}(x')\delta(n^\rho x'_\rho)\mathrm{d}^4x' = H, \tag{23}$$

or equivalently

$$[\mathcal{A}H]_- = 0. \tag{24}$$

This can be summarized:

Proposition 3: Under *PCT*-transformation the action *W* and the Hamiltonian *H* of the spinor field defined by the Lagrangian (1) are invariant.

Hence, with any solution of the corresponding Schroedinger equation its *PCT*-transform must be a solution too. If in quantum field theory the Schroedinger equation is replaced by the algebraic Schroedinger representation, instead of quantum mechanical wave functions state functionals are used to describe the state of the quantum system under consideration. The latter state functionals are solutions of corresponding functional equations, and inspite of this complication the same consequences as in quantum mechanics result from (31). For brevity we do not discuss this in detail in order to concentrate on the investigation of the effect of *PCT*-transformations on GBBW-equations, the solutions of which are the basis for constructing representations of such state functionals.

3. Representation of Formal Charge Symmetry

The spinor field characterized by the Lagrangian density (1) is not only forminvariant under the combined *PCT*-transformations, but also under *C*-transformations. This can be easily concluded from the deduction of (12) and (16), which gives for $\mathcal{C}$ only

$$\mathcal{C}\mathcal{L}(x)\mathcal{C}^{-1} = \mathcal{L}(x). \tag{25}$$

So it is interesting to analyze the physical meaning of this invariance or symmetry, respectively.

In [13] the probability interpretation of the quantum theory generated by (1) and (2) was discussed and deduced. From these considerations it follows that the basic (auxiliary) Dirac fields $\psi_{A\alpha i}(x)$ cannot be associated with observable quantities. Hence any charge which may be attached to the operation of charge conjugation (4) must be necessarily unobservable too. So these charges remain formal and have no physical meaning. Nevertheless this charge conjugation symmetry exists and it is the question how to incorporate it into the evaluation of this theory.

This point of view is supported if one considers the transformation properties of the basic fields under Lorentz-transformations which are given by

$$\begin{aligned} \psi'(x') &= S(a)\psi(a^{-1}x'), \\ \bar{\psi}'(x') &= \bar{\psi}(a^{-1}x')S^{-1}(a), \end{aligned} \tag{26}$$

and which are different.

So we have to look for a representation of the basic field operators which make this hidden symmetry manifest. We replace the adjoint spinor fields by charge conjugated spinor fields, which are defined by, see [14], Eq. (2.98)

$$\psi^c = C\bar{\psi}^T \tag{27}$$

or eqivalently by $\psi^c_{A\alpha i}(x) = C_{\alpha\beta}\bar{\psi}_{A\beta i}(x)$, see [11], Eq. (2.65), where with respect to the application of charge conjugation transformations the form (27) has to be preferred.

If this operation is applied to $\psi$and ψ^c from (4) and (27) it follows

$$\mathcal{C}\psi\mathcal{C}^{-1} = \psi^c; \quad \mathcal{C}\psi^c\mathcal{C}^{-1} = \psi, \tag{28}$$

Then it can be concluded from (27) that under Lorentz transformations the charge conjugated spinor transforms isomorphic to the spinor field, see [7], Eq. (4.14), p. 285 or [3], Eq. (15)

$$\psi^c(x')' = S(a)\psi^c(a^{-1}x'), \tag{29}$$

and this property allows to introduce superspinors by the definition

$$\psi_{A\Lambda\alpha i}(x) = \begin{pmatrix} \psi_{A\alpha i}(x); & \Lambda = 1 \\ \psi^c_{A\alpha i}(x); & \Lambda = 2 \end{pmatrix}. \tag{30}$$

Finally, in a last step we combine the indices A and Λ into one superindex κ, which is defined by

$$\kappa = \begin{pmatrix} 1 \equiv & \Lambda = 1, & A = 1 \\ 2 \equiv & \Lambda = 1, & A = 2 \\ 3 \equiv & \Lambda = 2, & A = 1 \\ 4 \equiv & \Lambda = 2, & A = 2 \end{pmatrix}. \tag{31}$$

The corresponding equations of motion which can be drived from (1) are then given by, see [11], Eq. (2.69)

$$\begin{aligned} &(D^\mu_{Z_1Z_2}\partial_\mu - m_{Z_1Z_2})\psi_{Z_2}(x) \\ &\quad = U_{Z_1Z_2Z_3Z_4}\psi_{Z_2}(x)\psi_{Z_3}(x)\psi_{Z_4}(x) \end{aligned} \tag{32}$$

with

$$\begin{aligned} D^\mu_{Z_1Z_2} &:= i\gamma^\mu_{\alpha_1\alpha_2}\delta_{\kappa_1\kappa_2}\delta_{i_1i_2}, \\ m_{Z_1Z_2} &:= m_{i_1}\delta_{\alpha_1\alpha_2}\delta_{\kappa_1\kappa_2}\delta_{i_1i_2}, \\ U_{Z_1Z_2Z_3Z_4} &:= \sum_h g\lambda_{i_1}B_{i_2i_3i_4}v^h_{\alpha_1\alpha_2}(v^hC)_{\alpha_3\alpha_4} \\ &\qquad \cdot\ \delta_{\kappa_1\kappa_2}[\gamma^5(1-\gamma^0)]_{\kappa_3\kappa_4}, \\ B_{i_2i_3i_4} &:= 1, \quad i_2, i_3, i_4 = 1, 2, 3. \end{aligned} \tag{33}$$

With indexing (30), i. e. referred to A and Λ, definitions (33) can be found in [11], Eq. (2.70).

In the new set of indices (31) the discrete *PCT*-transformations for spinor field operators are represented in the following way:

$$\begin{aligned} \mathcal{A}\psi_{A1\alpha i}(x)\mathcal{A}^{-1} &= -(\gamma^5\gamma^0C)_{\alpha\beta}\psi_{A2\beta i}(-x), \\ \mathcal{A}\psi_{A2\alpha i}(x)\mathcal{A}^{-1} &= (\gamma^5\gamma^0C)_{\alpha\beta}\psi_{A1\beta}(-x), \end{aligned} \tag{34}$$

or equivalently

$$\mathcal{A}\psi_{\kappa\alpha i}(x)\mathcal{A}^{-1} = (\gamma^5\gamma^0)_{\kappa\kappa'}(\gamma^5\gamma^0C)_{\alpha\alpha'}\psi_{\kappa'\alpha' i}(-x) \tag{35}$$

and

$$\mathcal{C}\psi_{\kappa\alpha i}(x)\mathcal{C}^{-1} = \gamma^5_{\kappa\lambda}\psi_{\lambda\alpha i}(x), \tag{36}$$

and it is our aim to apply these operations to the equation of motion (32). But before doing so, still another hidden symmetry has to be made manifest. As the vanishing of the anticommutator (8) holds for the single components of the spinor fields $\bar{\Psi}(x)$and $\Psi(x)$, this result is independent of transpositions. Hence the anticommutator of $\Psi^c(x)$and $\Psi(x)$vanishes too, and this means that all fields on the right hand side of (32) anticommute. Therefore the product of these fields act as projector on the vertex $U_{Z_1Z_2Z_3Z_4}$ which antisymmetrizes this vertex in its indices Z_2, Z_3, Z_4. Therefore from the beginning this symmetry has to be incorporated into the formalism in order to make it manifest.

The antisymmetrized vertex can be defined by

$$U_{Z_1Z_2Z_3Z_4} = g\lambda_{i_1} B_{i_2i_3i_4} V^{\alpha_1\alpha_2\alpha_3\alpha_4}_{\kappa_1\kappa_2\kappa_3\kappa_4} \quad (37)$$

with

$$V^{\alpha_1\alpha_2\alpha_3\alpha_4}_{\kappa_1\kappa_2\kappa_3\kappa_4} := \sum_h [v^h_{\alpha_1\alpha_2}\delta_{\kappa_1\kappa_2}(v^hC)_{\alpha_3\alpha_4}\gamma^5_{\kappa_3\kappa_4} - v^h_{\alpha_1\alpha_3}\delta_{\kappa_1\kappa_3}(v^hC)_{\alpha_2\alpha_4}\gamma^5_{\kappa_2\kappa_4} - v^h_{\alpha_1\alpha_4}\delta_{\kappa_1\kappa_4}(v^hC)_{\alpha_3\alpha_2}\gamma^5_{\kappa_3\kappa_2}], \quad (38)$$

where (38) is antisymmetrized in the index sets $(\alpha_2, \kappa_2), (\alpha_3, \kappa_3), (\alpha_4, \kappa_4)$.

Using this vertex expression, equation (32) can be written:

$$(i\gamma^\mu_{\alpha_1\alpha_2}\partial_\mu - m_{i_1}\delta_{\alpha_1\alpha_2})\delta_{\kappa_1\kappa_2}\delta_{i_1i_2}\psi_{\kappa_2\alpha_2i_2}(x) = g\lambda_{i_1} V^{\alpha_1\alpha_2\alpha_3\alpha_4}_{\kappa_1\kappa_2\kappa_3\kappa_4} \Psi_{\kappa_2\alpha_2}(x)\Psi_{\kappa_3\alpha_3}(x)\Psi_{\kappa_4\alpha_4}(x). \quad (39)$$

Obviously the transformation (36) is simpler than the transformation (35). Thus we start with application of (36) on Eq. (39). This leads to

Proposition 4: Equation (39) is forminvariant under the charge conjugation tranformation (36).

Proof: Application of (36) to (39) means replacement of ψ and Ψ in (39) by the corresponding right hand side of (36), i.e., by $\gamma^5\psi$ or $\gamma^5\Psi$, respectively: Direct evaluation shows that γ^5 can be factored out and eliminated, restoring the original equation. $\diamondsuit$

Proposition 5: Equation (39) is forminvariant under the *PCT*-transformations (35).

Proof: Application of the operator transformation (35) to Eq. (39). $\diamondsuit$

4. *PCT*-invariance of one-body GBBW-equations

GBBW-equations for *n*-parton states originate from a general GBBW-functional equation by projection, see [1], Eq. (1). From this equation one obtains in the one-parton sector the following GBBW-equation in configuration space:

$$[D^\mu_{Z_1X_1}\partial_\mu(x_1) - m_{Z_1X_1}]\varphi_{X_1}(x_1) = 0, \quad (40)$$

i. e., GBBW-equations for $n = 1$ are ordinary Dirac equations. We treat this problem in order to illustrate the general procedure for the most simple case.

In the preceding sections it was demonstrated that the action *W* as well as the Hamiltonian *H* of the nonlinear spinor field Lagrangian (1) are invariant under *PCT*-transformations, i.e., they commute with the corresponding *PCT*-operator $\mathcal{A}$. Furthermore, the field equations of this model are forminvariant under *PCT*-transformations and symmetric with respect to the charge conjugation operation.

From these results it follows that these operations reflect the *PCT*- and the *C*-invariance of the spinor field system, and that thus the eigenstates of *H* must be transformed into other eigenstates of *H* under the action of $\mathcal{A}$ and $\mathcal{C}$.

But, in quantum field theory eigenstates of *H* and of other constraints cannot be directly calculated. Owing to an infinite number of inequivalent representations of the field algebra, one is forced to calculate only the projections of appropriate basis systems on eigenstates $|a\rangle$. The latter basis systems are generated by application of monomials of field operators on a hypothetical groundstate $\langle 0|$(*GNS*-construction), see [11].

The GBBW-equations are part of this formalism, see [11]. And although their solutions are by definition and construction no solutions of the complete theory, their transformation properties must correspond to those of the covariant matrix elements of this theory.

This means that the GBBW-equations themselves must be forminvariant under the symmetry groups of the complete theory in order to guarantee the correct transformation properties of their solutions.

In the case of solutions for $n = 1$, the corresponding matrix elements of the complete theory read

$$\tau^{(1)}_Z(x) := \langle 0|\psi_Z(x)|a\rangle, \quad (41)$$

and the solutions of (40) must be transformed in agreement with the transformation properties of the matrix elements (41). For the Poincaré group the latter transformation property follows directly from the fact that the solutions of (40) are Dirac spinors. The same argument holds for the algebraic groups. With respect to the discrete groups the transformation (35) is an antiunitary transformation, and the transformation rules for scalar products, i.e., matrix elements are given by the following relation, see [6], Eq. (8.84):

$$\langle a|\mathcal{O}|b\rangle = \langle a'|\mathcal{O}'|b'\rangle^* = \langle b'|\mathcal{O}'^{+}|a'\rangle, \tag{42}$$

where the primed quantities are the transformed quantities. Application of (42) to (41) gives

$$\begin{aligned}\langle 0|\psi_{\kappa\alpha i}(x)|a\rangle &\\ &= (\gamma^5\gamma^0)_{\kappa\kappa'}(\gamma^5\gamma^0 C)_{\alpha\alpha'}\langle 0|\psi_{\kappa'\alpha' i}(-x)|a'\rangle^* \qquad (43)\\ &= (\gamma^5\gamma^0)_{\kappa\kappa'}(\gamma^5\gamma^0 C)_{\alpha\alpha'}\langle a'|\psi_{\kappa'\alpha' i}(-x)^+|0\rangle,\end{aligned}$$

and with

$$\psi_{\kappa\alpha i}(x')^+ = -(\gamma^0 C)_{\alpha\alpha'}\gamma^5_{\kappa\kappa'}\psi_{\kappa'\alpha' i}(x\prime) \tag{44}$$

the second line of (43) goes over into

$$\langle 0|\psi_{\kappa\alpha i}(x)|a\rangle = \gamma^0_{\kappa\kappa'}\gamma^5_{\alpha\alpha'}\langle a'|\psi_{\kappa'\alpha' i}(x')|0\rangle. \tag{45}$$

Now, if the solutions of (40) have to obey the same transformation rules, this leads either to

$$\varphi_{\kappa\alpha i}(x) = (\gamma^5\gamma^0)_{\kappa\kappa'}(\gamma^5\gamma^0 C)_{\alpha\alpha'}\chi_{\kappa'\alpha' i}(-x)^* \tag{46}$$

or to

$$\varphi_{\kappa\alpha i}(x) = \gamma^0_{\kappa\kappa'}\gamma^5_{\alpha\alpha'}\xi_{\kappa'\alpha' i}(-x). \tag{47}$$

Substitution of (46) into (40) gives

$$\begin{aligned}&(i\gamma^\mu_{\alpha_1\alpha_2}\partial_\mu(x) - m_{i_1}\delta_{\alpha_1\alpha_2})\delta_{\kappa_1\kappa_2}\delta_{i_1 i_2} \qquad (48)\\ &\quad\cdot(\gamma^5\gamma^0)_{\kappa_2\kappa_2'}(\gamma^5\gamma^0 C)_{\alpha_2\alpha_2'}\chi_{\kappa_2'\alpha_2' i_2}(-x)^* = 0,\end{aligned}$$

which after elimination of the transformation matrices leads to

$$\begin{aligned}&(i^*\gamma^{\mu *}_{\alpha_1\alpha_2}\partial_\mu(-x) - m_{i_1}\delta_{\alpha_1\alpha_2})\\ &\quad\cdot\delta_{\kappa_1\kappa_2}\delta_{i_1 i_2}\chi_{\kappa_2\alpha_2 i_2}(-x)^* = 0,\end{aligned} \tag{49}$$

or after complex conjugation with $x' = -x$

$$(i\gamma^\mu_{\alpha_1\alpha_2}\partial_\mu(x') - m_{i_1}\delta_{\alpha_1\alpha_2})\delta_{\kappa_1\kappa_2}\delta_{i_1 i_2}\chi_{\kappa_2\alpha_2 i_2}(x') = 0, \tag{50}$$

i. e., under the *PCT*-transformation (46) the one-parton GBBW-equation (40) is forminvariant. This means: with $\varphi(x)$also

$$\chi_{\kappa\alpha i}(x) = (\gamma^5\gamma^0)^{-1}_{\kappa\kappa'}(\gamma^5\gamma^0 C)^{-1}_{\alpha\alpha'}\varphi_{\kappa'\alpha' i}(-x)^* \tag{51}$$

is a solution of (40).

The same procedure can be performed with the transformation (47) by substituting it into (40). In this case one obtains

$$(i\gamma^\mu_{\alpha_1\alpha_2}\partial_\mu(x') - m_{i_1}\delta_{\alpha_1\alpha_2})\delta_{\kappa_1\kappa_2}\delta_{i_1 i_2}\xi_{\kappa_2\alpha_2 i_2}(x') = 0, \tag{52}$$

i. e., under the *PCT*-transformation (47) the one-parton GBBW-equation (40) is forminvariant too.

Hence with $\varphi(x)$also

$$\xi_{\kappa\alpha i}(x) = \gamma^0_{\kappa\kappa'}\gamma^5_{\alpha\alpha'}\varphi_{\kappa'\alpha' i}(-x) \tag{53}$$

is a solution of (40) too.

As the transformation laws for interacting fields and for free fields are the same, without loss of generality the wave functions φ can be identified with matrix elements of free spinor fields $\psi^f(x)$ in the case under consideration. Then one can study in detail the physical meaning of the various matrix elements which are simultaneously solutions of (40). But for brevity we suppress this discussion.

5. *PCT*-invariance of Two-body GBBW-equations

From the functional equation in [1], Eq. (1), by a corresponding projection the two-body GBBW-equations can be derived. They read

$$\begin{aligned}&(i\gamma^\mu\partial_\mu(x_1) - m_{i_1})_{\beta_1\alpha_1}\delta_{\rho_1\kappa_1}\delta_{i_1 i_2}\varphi^{\kappa_1\rho_2}_{\alpha_1\beta_2}(x_1,x_2)_{i_1 i_2}\\ &= -3\lambda_{i_1}\frac{1}{2}gV^{\rho_1\kappa_2\kappa_3\kappa_4}_{\beta_1\alpha_2\alpha_3\alpha_4}\sum_i F^{\kappa_4\rho_2}_{\alpha_4\beta_2}(x_1-x_2)_{ii_2} \qquad (54)\\ &\qquad\cdot\ \hat\varphi^{\kappa_2\kappa_3}_{\alpha_2\alpha_3}(x_1,x_1)\end{aligned}$$

where $\hat\varphi$ is the regularized solution. And also in this case the transformation properties of the solutions of these equations must correspond to those of the matrix elements of the complete theory.

In the two-body case these matrix elements are defined by

$$\tau^{(2)}_{Z_1 Z_2}(x_1,x_2) := \langle 0|T\psi_{Z_1}(x_1)\psi_{Z_2}(x_2)|a\rangle, \tag{55}$$

where *T* means time ordering which is defined by

$$\begin{aligned}\tau^{(2)}_{Z_1 Z_2}(x_1,x_2) &= \Theta(t_1-t_2)\langle 0|\psi_{Z_1}(x_1)\psi_{Z_2}(x_2)|a\rangle\\ &\quad - \Theta(t_2-t_1)\langle 0|\psi_{Z_2}(x_2)\psi_{Z_1}(x_1)|a\rangle.\end{aligned} \tag{56}$$

If a *PCT*-transformation is applied to the matrix elements in (56), formula (42) can be used to relate the original matrix elements to the transformed ones.

In order to describe the transformed process by time ordered matrix elements too (for instance the *S*-matrix), the property of time ordering must be conserved. This is possible if (42) is reduced to the antiunitary transformation law

$$\langle a|\mathcal{O}|b\rangle = \langle b'|(\mathcal{O}')^+|a'\rangle, \tag{57}$$

and with $x' = -x$ one obtains from (56) by application of (57)

$$\begin{aligned}\tau^{(2)}_{Z_1Z_2}(x_1,x_2) &= \Theta(t'_2-t'_1)\langle a'|\psi'_{Z_2}(x'_2)^+\psi'_{Z_1}(x'_1)^+|0\rangle \\ &- \Theta(t'_1-t'_2)\langle a'|\psi'_{Z_1}(x'_1)^+\psi'_{Z_2}(x'_2)^+|0\rangle.\end{aligned} \tag{58}$$

In accordance with (35) and (44) the Hermitian conjugate of the transformed spinor ψ' can be expressed by

$$\psi'_{\kappa\alpha i}(x')^+ = \gamma^0_{\kappa\kappa'}\gamma^5_{\alpha\alpha'}\psi_{\kappa'\alpha' i}(x'), \tag{59}$$

and this leads to the transformation law for matrix elements under *PCT*-transformations:

$$\begin{aligned}\tau^{(2)}_{Z_1Z_2}(x_1,x_2)_{i_1i_2} &= -\gamma^0_{\kappa_1\kappa'_1}\gamma^5_{\alpha_1\alpha'_1}\gamma^0_{\kappa_2\kappa'_2}\gamma^5_{\alpha_2\alpha'_2} \\ &\quad\cdot\ \tau^{\kappa'_1\kappa'_2}_{\alpha'_1\alpha'_2}(x'_1,x'_2)'_{i_1i_2}.\end{aligned} \tag{60}$$

Hence φ must obey the same transformation law, i. e.,

$$\begin{aligned}\varphi_{Z_1Z_2}(x_1,x_2)_{i_1i_2} &= -\gamma^0_{\kappa_1\kappa'_1}\gamma^5_{\alpha_1\alpha'_1}\gamma^0_{\kappa_2\kappa'_2}\gamma^5_{\alpha_2\alpha'_2} \\ &\quad\cdot\ \varphi^{\kappa'_1\kappa'_2}_{\alpha'_1\alpha'_2}(x'_1,x'_2)'_{i_1i_2},\end{aligned} \tag{61}$$

in order to be compatible with the representation of states of the complete theory.

Furthermore the propagator *F* can be expressed by the vacuum expectation value of free spinor fields

$$F_{Z_1Z_2}(x_1,x_2) = \langle 0|T\psi^f_{Z_1}(x_1)\psi^f_{Z_2}(x_2)|0\rangle, \tag{62}$$

and since free and interacting spinor fields transform in the same way and the vacuum is invariant under *PCT*-transformations (by definition), the identity

$$\begin{aligned}F^{\kappa_1\kappa_2}_{\alpha_1\alpha_2}(x_1,x_2)_{i_1i_2} &= -\gamma^0_{\kappa_1\kappa'_1}\gamma^5_{\alpha_1\alpha'_1}\gamma^0_{\kappa_2\kappa'_2}\gamma^5_{\alpha_2\alpha'_2} \\ &\quad\cdot\ F^{\kappa'_1\kappa'_2}_{\alpha'_1\alpha'_2}(x'_1,x'_2)_{i_1i_2}\end{aligned} \tag{63}$$

must hold, which can be easily verified by substitution of the explicit expression, see [1], Eq. (10) or [11], Eq. (3.115).

If (61) and (63) are substituted in (54), the indices of the matrices $\gamma^0_{\rho_2\rho'_2}\gamma^5_{\beta_2\beta'_2}$ are spectator indices, and these matrices can be eliminated. This yields the equation

$$\begin{aligned}&-(i\gamma^\mu\partial_\mu(x_1)-m_{i_1})_{\beta_1\alpha_1}\delta_{\rho_1\kappa_1}\delta_{i_1i_2}\gamma^0_{\kappa_1\kappa'_1}\gamma^5_{\alpha_1\alpha'_1} \\ &\quad\cdot\ \varphi^{\kappa'_1\rho'_2}_{\alpha'_1\beta'_2}(x'_1,x'_2)'_{i_1i_2} \\ &= -3\lambda_{i_1}\frac{1}{2}gV^{\rho_1\kappa_2\kappa_3\kappa_4}_{\beta_1\alpha_2\alpha_3\alpha_4}\gamma^0_{\kappa_2\kappa'_2}\gamma^0_{\kappa_3\kappa'_3}\gamma^0_{\kappa_4\kappa'_4}\gamma^5_{\alpha_2\alpha'_2}\gamma^5_{\alpha_3\alpha'_3} \\ &\qquad\cdot\ \gamma^5_{\alpha_4\alpha'_4}\sum_i F^{\kappa'_4\rho'_2}_{\alpha'_4\beta'_2}(x'_1,x'_2)_{ii_2}\hat{\varphi}^{\kappa'_2\kappa'_3}_{\alpha'_2\alpha'_3}(x'_1,x'_2)'.\end{aligned} \tag{64}$$

With $v^1 := \mathbf{1}$, $v^2 := i\gamma^5$ and (38) one can prove the relation

$$\begin{aligned}&V^{\rho_1\kappa_2\kappa_3\kappa_4}_{\beta_1\alpha_2\alpha_3\alpha_4}\gamma^0_{\kappa_2\kappa'_2}\gamma^0_{\kappa_3\kappa'_3}\gamma^0_{\kappa_4\kappa'_4}\gamma^5_{\alpha_2\alpha'_2}\gamma^5_{\alpha_3\alpha'_3}\gamma^5_{\alpha_4\alpha'_4} \\ &= -\gamma^0_{\rho_1\kappa_1}\gamma^5_{\beta_1\alpha_1}V^{\kappa_1\kappa'_2\kappa'_3\kappa'_4}_{\alpha_1\alpha'_2\alpha'_3\alpha'_4}\end{aligned} \tag{65}$$

and the additional relation

$$\begin{aligned}&(i\gamma^\mu\partial_\mu(x_1)-m_{i_1})_{\beta_1\alpha_1}\delta_{\rho_1\kappa_1}\gamma^0_{\kappa_1\kappa'_1}\gamma^5_{\alpha_1\alpha'_1} \\ &= \gamma^0_{\rho_1\kappa_1}\gamma^5_{\beta_1\alpha_1}(i\gamma^\mu\partial_\mu(x')-m_{i_1})_{\alpha_1\alpha'_1}\delta_{\kappa_1\kappa'_1}.\end{aligned} \tag{66}$$

Substitution of (65) and (66) and eliminating $\gamma^0_{\rho_1\kappa_1}\gamma^5_{\beta_1\alpha_1}$ finally gives

$$\begin{aligned}&(i\gamma^\mu\partial_\mu(x'_1)-m_{i_1})_{\beta_1\alpha_1}\delta_{\rho_1\kappa_1}\varphi^{\kappa_1\rho_2}_{\alpha_1\beta_2}(x'_1,x'_2)'_{i_1i_2} \\ &= -3\lambda_{i_1}\frac{1}{2}gV^{\rho_1\kappa_2\kappa_3\kappa_4}_{\beta_1\alpha_2\alpha_3\alpha_4}\sum_i F^{\kappa_4\rho_2}_{\alpha_4\beta_2}(x'_1-x'_2)_{ii_2} \\ &\qquad\cdot\ \hat{\varphi}^{\kappa_2\kappa_3}_{\alpha_2\alpha_3}(x'_1,x'_1)',\end{aligned} \tag{67}$$

if the primes on the indices are suppressed.

These results can be summarized in the following way:

Proposition 6: The transformation law of the solutions of the two-body GBBW-equations is given by (61), and the GBBW-equations themselves are forminvariant unter *PCT*-transformations.

The extension of this proof to many body GBBW-equations runs along the same lines as the treatment of the two-body case and thus needs no further explicit discussion.

6. Algebraic Formulation of Discrete Operations

The characteristic feature of the algebraic formalism is that it admits an infinite number of inequivalent state spaces for an explicit representation of the theory. In the preceding sections we assumed the existence of the operators $\mathcal{P}$, $\mathcal{C}$, $\mathcal{T}$ and their inverses in state space without giving an explicit construction. Therefore concerning the existence of these operators the question arises whether they can be defined independently of state representations in order to fit into the algebraic formalism.

In the literature such operators are explicitly constructed in Fock space with Dirac vacuum, see [5] and [12]. We are going to show that these definitions can be generalized to a form which is valid in the algebraic formalism for the model under consideration. The basic relations of the action of these operators are given by (3),(4),(5). If by explicit construction of $\mathcal{P}$, $\mathcal{C}$, $\mathcal{T}$ these actions can be realized, then our problem is solved, because all other proofs are based on these relations.

We start with the parity operation. In constructing a parity operator for the spinor field theory defined by (1) and (2), we solve this problem by relating this operator to the conventional algebraic formulation of the parity operator given in [5], Eq. (15.93), which is taken for granted.

Proposition 7: The algebraic expression of the parity operator is given by

$$\mathcal{P} := \exp\{-i\frac{1}{2}\pi \sum_{s,A,j} \int \mathrm{d}^3k \lambda_j^{-1} [c_{Aj}(\mathbf{k},s)^+ c_{Aj}(\mathbf{k},s) - c_{Aj}(\mathbf{k},s)^+ c_{Aj}(-\mathbf{k},s) + \mathrm{d}_{Aj}(\mathbf{k},s)^+ d_{Aj}(\mathbf{k},s) + d_{Aj}(\mathbf{k},s)^+ d_{Aj}(-\mathbf{k},s)]\}, \tag{68}$$

and the Hamiltonian of the spinor field is invariant under the application of this operator.

Proof: Choosing the special vector n^ρ = (1, 0, 0, 0) from (21) it follows that *H* depends only on $\psi(\mathbf{r},0)$ and $\bar{\psi}(\mathbf{r},0)$. Hence the invariance of *H* can be proven if the equations (3) hold for $t=0$ only. We first study this special case.

For ψ we use the expansion

$$\psi_{A\alpha i}(\mathbf{r},0) = \sum_s \int \mathrm{d}^3k (2\pi)^{-\frac{3}{2}} \left(\frac{m_i}{E_k}\right)^{\frac{1}{2}} \cdot [u(\mathbf{k},s)_{\alpha i} c_{Ai}(\mathbf{k},s) e^{i\mathbf{kr}} + v(\mathbf{k},s)_{\alpha i} d_{Ai}(\mathbf{k},s)^+ e^{-i\mathbf{kr}}], \tag{69}$$

where the various symbols are defined by the corresponding expansion of the free Dirac field at $t=0$, see [5] and [12]. The inverse relations read

$$c_{Ai}(\mathbf{k},s) = \left(\frac{m_i}{E_k}\right)^{\frac{1}{2}} \int \mathrm{d}^3 r e^{-i\mathbf{kr}} \bar{u}(\mathbf{k},s)_{\alpha i} \cdot \gamma^0_{\alpha\beta} \psi_{A\beta i}(\mathbf{r},0),$$
$$\mathrm{d}_{Ai}(\mathbf{k},s)^+ = \left(\frac{m_i}{E_k}\right)^{\frac{1}{2}} \int \mathrm{d}^3 r e^{i\mathbf{kr}} \bar{v}(\mathbf{k},s)_{\alpha i} \cdot \gamma^0_{\alpha\beta} \psi_{A\beta i}(\mathbf{r},0). \tag{70}$$

Let us now consider the expansion (69) as an abstract algebraic transformation without reference to a specific state space. Then the operators (70) can be defined in a general algebraic way by their anticommutation relations independently of any Fock space interpretation. By direct calculation one obtains

$$[c_{Ai}(\mathbf{k},s) c_{A'i'}(\mathbf{k}',s')^+]_+ = \lambda_i \delta_{ii'} \delta_{AA'} \delta_{ss'} \delta(\mathbf{k}-\mathbf{k}'),$$
$$[\mathrm{d}_{Ai}(\mathbf{k},s) d_{A'i'}(\mathbf{k}',s')^+]_+ = \lambda_i \delta_{ii'} \delta_{AA'} \delta_{ss'} \delta(\mathbf{k}-\mathbf{k}'). \tag{71}$$

In Fock space the operators *c* and *d* annihilate the vacuum, whereas in more general state spaces this property cannot be maintained. i. e., $c|0\rangle \neq 0, \quad d|0\rangle \neq 0$ holds, which constitutes the essential difference between Fock spaces and other representation spaces. But this difference does not influence the algebraic relations which are based on the anticommutation relations.

If the renormalized operators c' and d' are introduced by

$$c_{Ai}(\mathbf{k},s) = \lambda_i^{\frac{1}{2}} c'_{Ai}(\mathbf{k},s), \quad d_{Ai}(\mathbf{k},s) = \lambda_i^{\frac{1}{2}} d'_{Ai}(\mathbf{k},s), \tag{72}$$

they satisfy the anticommutation relations

$$[c'_{Ai}(\mathbf{k},s) c'_{A'i'}(\mathbf{k}',s')^+]_+ = \delta_{ii'} \delta_{AA'} \delta_{ss'} \delta(\mathbf{k}-\mathbf{k}'),$$
$$[\mathrm{d}'_{Ai}(\mathbf{k},s) d'_{A'i'}(\mathbf{k}',s')^+]_+ = \delta_{ii'} \delta_{AA'} \delta_{ss'} \delta(\mathbf{k}-\mathbf{k}'). \tag{73}$$

Substitution of these operators in (68) yields

$$\mathcal{P}(c,d) = \mathcal{P}'(c',d'), \tag{74}$$

where the primed operators completely correspond to the conventional definitions in [5] and [12]. In [12] it is demonstrated that the latter operators lead to the transformation relations

$$\begin{aligned} \mathcal{P}' c'_{Ai}(\mathbf{k},s)\mathcal{P}'^{-1} &= c'_{Ai}(-\mathbf{k},s), \\ \mathcal{P}' d'_{Ai}(\mathbf{k},s)^{+}\mathcal{P}'^{-1} &= -d'_{Ai}(-\mathbf{k},s)^{+}, \end{aligned} \tag{75}$$

and corresponding relations for the Hermitian conjugates.

Furthermore, in [5] it is shown that from (75) the transformation law (3) for $t = 0$ can be derived. Owing to the dependence of H on the field operators of the hyperplane $t = 0$, relation (3) for $t = 0$ suffices to prove that

$$\mathcal{P} H \mathcal{P}^{-1} \equiv H \tag{76}$$

holds for the spinor field Hamiltonian (19). For brevity this is not explicitly demonstrated.

Observing that $\psi(\mathbf{r},t) = \exp(-iHt)\psi(\mathbf{r},0)\exp(iHt)$ describes the time dependence in the Heisenberg picture, one easily verifies that (3) must be valid. $\diamondsuit$

Similar considerations with respect to the existence of algebraic representations of discrete operations in more general state spaces can be done for charge conjugation and time inversion, where the Fock space formulations are given in [5]. For brevity we suppress the corresponding derivations.

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